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ANALYTICAL THEORY OF REGENERATIVE HEAT TRANSFER AND STUDY OF ITS THERMODYNAMIC EFFICIENCY

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An analytical theory of the cyclic process of regenerative heat transfer is developed. Introduction of operator-valued functions of integral operators allowed the authors to formulate an equation for determination of periodic solutions. A solution of this equation is found in a linear approximation with respect to switching times. A functional – exergic efficiency – is introduced on periodic solutions. The analytical solution obtained made it possible to study the dependence of the introduced criterion on the parameters of the system and to describe the time mode of the process in which the thermodynamic efficiency is maximum. Comparison of the analytical theory and a numerical analysis of the problem showed that upon choosing an operating mode of the apparatus that fits the maximum values of the introduced criterion, one can use results of the analytical solution at all admissible values of the switching times.

Regenerative heat transfer through a layer of solid-phase heat-accumulating material with switching of gas flows is a rather common technological operation. Additional interest in this process was generated by the prospects for using it as one of the structural elements of adsorption cycles of heat pumps and refrigerators [1-3].

Mathematical simulation of regenerative heat transfer is based, as a rule, on the use of the well-substantiated nonstationary one-dimensional equations of interphase heat transfer that are written below. However, we strongly feel the need for a compact analytical theory of the process that could allow us to distinguish periodic solutions of the nonstationary problem, which are of greatest importance in practice. Lack of this, in particular, makes a quantitative analysis of the thermodynamic efficiency of the system difficult.

It is obvious that, to find periodic solutions, one must be able to solve analytically the nonstationary problem of layer heating with arbitrary initial profiles of temperature. The history of the problem goes back to the work of W. Nusselt [4], in which a solution of this problem is obtained in the form of a convolution of the initial temperature distribution and combinations of Bessel functions of an imaginary argument. This makes it possible, in principle, to calculate step by step the steady-state mode of apparatus operation. However, it would be desirable to have a closed solution that describes the steady-state periodic mode of operation of the regenerator. It is easily seen that this approach to constructing a periodic solution leads to a system of integral equations.

A review of attempts to describe regenerative heat transfer under various additional simplifications by infinite series of convolution integrals with Bessel functions can be found in [5]. Even if we generalize solutions of this type to the case of unequal switching times, which is of interest in practice, such cumbersome expressions are unacceptable for solving posed problems of thermodynamic analysis of the system.

Besides the above methods of reduction of the problem to integral equations, we also use the Laplace transform to describe the temperature variation in the gas and solid phases from cycle to cycle with an initially uniform distribution of the temperature in the layer. With equal switching times, we can easily obtain recursion relations for the transforms of the initial functions. However, upon changing to the inverse transform, the solu-

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tion is obtained in the form of a cumbersome infinite series [6]. Generalization to the case of unequal times in this approach also involves great difficulties.

We developed an analytical method for obtaining periodic solutions of regenerative heat transfer. We succeeded in this thanks to the introduction of operator-valued functions of integral operators.

Mathematical physics knows examples of the use of polynomial and exponential types of operator-valued functions, but of differential operators [7]. In this paper we introduce an exponential function of integral operators. This made it possible to formulate an equation for determining a periodic mode of the process of regenerative heat transfer and to find the corresponding solution in a linear approximation.

Mathematical Model of a Regenerative Heat Exchanger. The problem of utilization of the thermal energy of a waste gas flow by means of a cyclic regenerative heat exchanger is considered. In the first stage, a hot gas heats a bulk layer of solid-phase material for a time τ_1. Then the heat exchanger is switched to the stage of heating a second flow of gas moving in the opposite direction. The second stage lasts for a time τ_2.

To describe the cyclic process of regenerative heat transfer, we use a one-dimensional mathematical model of nonstationary interphase heat transfer:

$$\frac{\partial\theta_1}{\partial\xi} = N(\varphi - \theta_1)\,,\ \theta_1(0,\tau) = 1\,,\ \frac{\partial\varphi}{\partial\tau} = \theta_1 - \varphi\,, \tag{1}$$

$$0 < \tau < \tau_1\,;\ \ 0 < \xi < 1\,;$$

$$\frac{\partial\theta_2}{\partial\xi} = NR(\theta_2 - \varphi)\,,\ \theta_2(1,\tau) = 0\,,\ \frac{\partial\varphi}{\partial\tau} = \theta_2 - \varphi\,,\ \ 0 < \tau < \tau_2\,;\ \ 0 < \xi < 1\,. \tag{2}$$

The equations of the model are written without regard for longitudinal heat conduction and under the condition that all thermophysical characteristics of both phases are constant within the working range of temperatures. Due to the large difference in the densities of the gas and solid phases, the process of heat transfer is assumed to be quasistationary with respect to the gas phase. The boundary conditions correspond to the conditions of a constant temperature of both the hot and cold gases entering the apparatus: $T_1(0, t) = T_{10}$ and $T_2(L, t) = T_{20}$; $T_{10} > T_{20}$.

After a certain number of cycles, the apparatus reaches the cyclic operating mode. The transient processes of its reaching this mode are not considered, and periodic solutions of the system of equations (1)-(2) are sought. For this purpose we introduce convolution integral operators S_1 and S_2 that act according to the following rule:

$$S_1\varphi = N\int_0^{\xi} d\chi\varphi(\chi)\exp(N(\chi - \xi))\,,\ \ S_2\varphi = NR\int_{\xi}^{1} d\chi\varphi(\chi)\exp(NR(\xi - \chi))\,. \tag{3}$$

It can be shown that the properties of these operators allow one to determine the product of the operators and power series and to introduce operator-valued analytical functions, which are integrated and differentiated term by term. As an example of these functions we give an exponent needed in the sequel:

$$\exp(-\tau S_i)\varphi = \sum_{n=0}^{\infty}\frac{(-\tau S_i)^n\varphi}{n!}\,;\ \ i = 1, 2\,. \tag{4}$$

Mathematical operations (differentiation, integration, etc.) in the left-hand side of the equality will be understood as term-by-term operations in the right-hand side.

The introduced operators (3) make it possible to reduce the dimensionality of the problem and to write an equation for finding a periodic solution of problem (1)-(2) that would include one unknown function. Actually, we express the functions θ_i, $i = 1, 2$ in terms of φ: $\theta_1 = \exp(-N\xi) + S_1\varphi$, $\theta_2 = S_2\varphi$. Then we assume that, as a result of a cyclic operation, the system reached the periodic mode. We assume that the temperature of the solid phase at the beginning of stage 1 has the distribution over the height of the apparatus $\varphi_0(\xi)$. It is for this unknown function that we obtain the required equation.

In stage 1, the evolution of the system is described by the sum of two functions: $\varphi(\xi, \tau) = H(\xi, \tau) + \Psi_1(\xi, \tau)$, where $H(\xi, \tau)$ is the solution of the inhomogeneous equation with zero initial conditions and $\Psi_1(\xi, \tau)$ is the solution of the homogeneous equation with the initial temperature distribution $\varphi_0(\xi)$. For $H(\xi, \tau)$ and $\Psi_1(\xi, \tau)$ we have

$$H(\xi, \tau) = \int_0^\tau \exp[\chi(S_1 - 1)]\, d\chi \exp(-N\xi) =$$

$$= \exp(-N\xi) \int_0^\tau \exp(-\chi)\, I_0(2\sqrt{N\eta\chi})\, d\chi\,,$$

$$\Psi_1(\xi, \tau) = \exp[(S_1 - 1)\tau]\, \varphi_0(\xi)\,.$$

Finally, we obtain that at the end of the first stage the temperature profile in the solid phase has the following form:

$$\varphi(\xi, \tau_1) = H(\xi, \tau_1) + \exp[(S_1 - 1)\tau_1]\, \varphi_0(\xi)\,.$$

The function $\varphi(\xi, \tau_1)$ is the initial condition for the evolution problem describing the stage of gas heating. Its solution is

$$\Psi_2(\xi, \tau) = \exp[(S_2 - 1)\tau]\, \varphi(\xi, \tau_1)\,.$$

Thus, for the unknown function $\varphi_0(\xi)$ we obtain the final equation

$$\varphi_0(\xi) = \exp[(S_2 - 1)\tau_2]\, (H(\xi, \tau_1) + \exp[(S_1 - 1)\tau_1]\, \varphi_0(\xi))\,. \tag{5}$$

It is obvious that quasiequilibrium conditions of regenerative heat transfer that provide a low level of losses of thermal exergy are attained with a sufficient volume of heat-transfer material; characterized by the parameter N, and relatively low values of the switching times τ_1 and τ_2. Therefore, we restrict ourselves to solution of Eq. (5) in a linear approximation with respect to τ_1 and τ_2.

In first order with respect to τ_1 and τ_2, Eq. (5) takes the following form:

$$0 = \tau_1 \exp(-N\xi) + \tau_1 (S_1 - 1)\, \varphi_0 + \tau_2 (S_2 - 1)\, \varphi_0\,. \tag{6}$$

We note that the times of the stages τ_1 and τ_2 enter this equation uniformly, i.e., the unknown function φ_0 depends on the ratio of these quantities. Therefore, for further calculations it is convenient to introduce the variable $\lambda = \tau_2/\tau_1$.

It is easy to verify that the function

$$\varphi_0(\xi) = \frac{\dfrac{\lambda(R+1)\exp(-\mu\xi)}{(\lambda+1)R} - \exp(-\mu)}{\dfrac{\lambda}{R} - \exp(-\mu)}\,, \quad \mu = \frac{(\lambda - R)N}{\lambda + 1}\,, \quad \lambda = \frac{\tau_2}{\tau_1} \tag{7}$$

is the solution of Eq. (6). At $\lambda/R = 1$ we obtain a degenerate solution – the linear profile

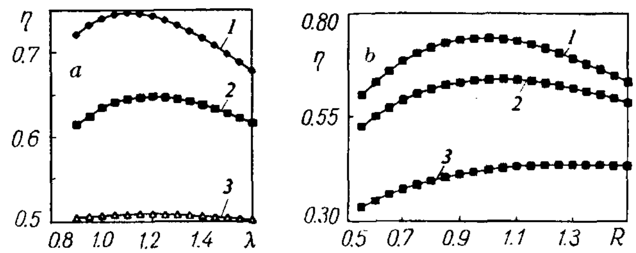

Fig. 1. Exergic criterion η vs. λ at R = 1 (a) and vs. R at λ = 1 (b): 1) N = 10, 2) 6.5, 3) 3.

$$\varphi_0(\xi) = \frac{1 + \frac{1}{NR} - \xi}{1 + \frac{1}{N} + \frac{1}{NR}}. \tag{8}$$

The analytical solution obtained allows one to rationally organize a numerical analysis of the problem and study the dependence of the criterion of thermodynamic ideality on the parameters of the system.

Criterion of Thermodynamic Ideality of the Cyclic Process. Let $\Delta H_1 = J_1 C_1 \int dt(T_1(z = L, t) - T_{10}) < 0$ be the change in the gas enthalpy in the first stage of the process and $\Delta H_2 = J_2 C_2 \int dt(T_2(z = 0, t) - T_{20} > 0$ be the total amount of heat used for heating the cold gas in the cyclic process in one period. The change in the entropy of the hot gas in the first stage is $\Delta S_1 = J_1 C_1 \int dt \ln (T_1(z = L, t)/T_{10} < 0$. The change in the entropy of the cold gas in the second stage is $\Delta S_2 = J_2 C_2 \int dt \ln (T_2(z = 0, t)/T_{20} > 0$. The decrease in the exergy of the hot gas in the first stage is defined by the quantity $\Delta E_1 = \Delta H_1 - T_0 \Delta S_1$ ($\Delta E_1 < 0$), and the change in the exergy of the heated-gas flow is $\Delta E_2 = \Delta H_2 - T_0 \Delta S_2$ ($\Delta E_2 > 0$).

We define the exergic efficiency η as the ratio of the increment in the exergy of the heated gas in the second stage ΔE_2 to the maximum possible removal of exergy from the hot gas in the first stage $\Delta E_1^{(*)}$, corresponding to the conditions where the gas temperature at the outlet from the apparatus in the first stage is equal to the gas temperature at the inlet in the second stage, $T_1(z = L, t) = T_{20}$:

$$\eta = \Delta E_2 / - \Delta E_1^{(*)},$$

where

$$-\Delta E_1^{(*)} = \sup(-\Delta E_1) = J_1 C_1 t_1 (T_{10} - T_{20} - T_0 \ln (T_{10}/T_{20})).$$

The criterion formulated differs from traditional criteria of exergic efficiency [8] and accounts for the special properties of the cyclic process. We emphasize that the criterion integrally reflects the effect of both sources of losses of thermal exergy in regenerative heat transfer: insufficiently complete removal of heat from the heating gas (incomplete cooling of the gas in the first stage) and incomplete transfer of thermal exergy from the heat-accumulating material to the heated gas.

Change in the Thermodynamic Efficiency as a Function of the Control Parameters of the System. First, we give the general laws of the change in the criterion η as a function of the parameters λ, R, and N that proceed from the above-obtained analytical solutions of the initial system of equations (1)-(2). Then, on the basis of a numerical analysis of this system, we show that the applicability range of the analytical solutions covers the range of the working parameters within which the values of the exergic efficiency are maximum.

As is shown in Fig. 1a, the dependence of the criterion η on λ at fixed R and N has a pronounced extremum character. At relatively small times of the second stage (small λ), the temperature profiles are convex; the main temperature difference and the generation of entropy (the exergy losses) are concentrated in the vicinity of the inlet of the heated flow. In contrast, at relatively small times of the first stage (large λ), the temperature profiles become concave; the main change of temperature and the generation of entropy are con-

TABLE 1. Maximum Values of the Criterion and the Values of λ Corresponding to Them as Functions of N at $R = 1$

N	3	6.5	10	40	70	100
η	0.43	0.65	0.75	0.93	0.96	0.97
λ	1.199	1.144	1.108	1.033	1.019	1.014

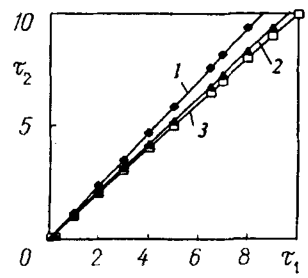

Fig. 2. Extrema of the exergic efficiency of regenerative heat transfer on the phase plane of the control parameters τ_1 and τ_2 at $R = 1$: 1) $N = 10$; 2) 100; 3) straight line $\lambda = R$.

centrated at the inlet of the hot flow. The situation where the temperature profiles approach a linear one (degenerate solution (8)) corresponds to an extremum of η. In this case, the generation of entropy is uniformly distributed over the entire height of the apparatus, but it is small due to a small motive force of the heat-transfer process.

Table 1 shows the manner in which the process of regenerative heat transfer should be conducted for the criterion of thermodynamic efficiency to remain maximum. The maximum values of the criterion and the values of λ corresponding to them with the number of transfer units N increasing from 3 to 100 are given. It is seen that the values of the criterion increase with N. Similar regularities hold true at other values of R.

Figure 1b illustrates the dependence of the criterion of efficiency η on the parameter R at fixed λ and N. The maximum value of the criterion increases with the number of transfer units.

It follows from the analytical solution that at a rather large number of transfer units, viz., when the inequalities $N >> 1$ and $NR >> 1$ are met, the maximum value of η for given N and R is reached for the condition $\lambda \approx R$, i.e., $C_1 J_1 t_1 \approx C_2 J_2 t_2$. Under these conditions, η is expressed by the asymptotic formula

$$\eta = 1 - \frac{(\gamma - \gamma_0)\left(\frac{1}{N} + \frac{1}{NR}\right)}{1 - \frac{\gamma_0 \ln \gamma}{\gamma - 1}}.$$

In the general case, beyond the region of validity of the analytical solutions, the temperature profiles and the criterion η depend not only on the ratio of the switching times λ but also on the values of τ_1 and τ_2 themselves. A numerical analysis of the initial system of equations (1)-(2) showed that at fixed N and R the graph of the function $\eta(\tau_1, \tau_2)$ is a surface that has the form of a crest. On the phase plane of τ_1 and τ_2, the coordinates of its peaks lie on an extremum that is a concave curve that coincides with the above-obtained asymptotic straight line $\tau_2 = \lambda\tau_1$ at the coordinate origin, but with increase in τ_1 it departs from this curve. When the conditions $\tau_1 \le N$ and $\tau_2 \le NR$, which correspond precisely to the region of relatively high values of η, are met, the deviations do not exceed several percent. Figure 2 shows that the extrema that are found by the numerical analysis of the initial system of equations (1)-(2) virtually do not differ from the extrema given in the table. Thus, we can draw the conclusion that in choosing an operating mode of the apparatus that corresponds to large values of the introduced criterion, we can use results of the analytical solution at all τ_1 and τ_2 for which $\tau_1 \le N$ and $\tau_2 \le NR$.

The numerical analysis of the problem also showed that in moving along an extremum, i.e., with increase in τ_1, the value of η decreases monotonically. This is associated with the increase in the generation of

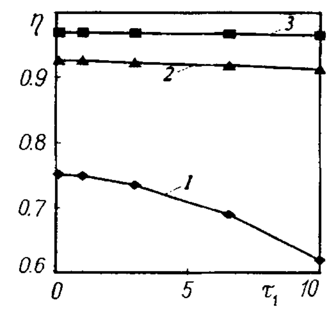

Fig. 3. Values of the exergic efficiency in moving along extrema at R = 1: 1) N = 10; 2) 40; 3) 100.

entropy in the system caused by the increase in the temperature gradients in switching from one stage to the other when the times of the stages are lengthened. When $\tau_1 > N$ and $\tau_2 > NR$, the value of η decreases sharply. Figure 3 shows the course of the decrease of the criterion of thermodynamic efficiency in moving along the corresponding extrema and illustrates the region on the phase plane of τ_1 and τ_2 where the analytical solution (7) can be used for evaluating the introduced criterion. It is seen that the smaller the value of N, the more the calculated value of η for a given τ_1 differs from the theoretical one, i.e., from η at $\tau_1 \to 0$.

Considering the decisive role of N in attaining a high thermodynamic efficiency, we note that for a specified volume of the apparatus and specified gas flows, the coefficient of heat transfer α should be increased in order to increase N. This can be achieved in systems with a hierarchically organized spatial structure of the bulk layer [9, 10].

NOTATION

C, heat capacity at constant pressure, J/(K·kg); $I_0(\xi)$, modified Bessel function; J, mass flow rate of the gas, kg/sec; M_s, mass of the solid phase, kg; L, height of the apparatus, m; T, absolute temperature, K; T_0, ambient temperature; t, time, sec; t_1, time of stage 1; t_2, time of stage 2; z, coordinate in the direction of gas motion in stage 1, m; α, volumetric coefficient of interphase heat transfer, J/(m^3·sec·K); ρ, density, kg/m^3. Dimensionless variables: $\theta_i = (T_i - T_{20})/(T_{10} - T_{20})$, i = 1, 2 for the gas temperature; $\varphi = (T_s - T_{20})/(T_{10} - T_{20})$ for the temperature of the solid phase; $N = \alpha M_s/(\rho_s C_1 J_1)$, number of transfer units for stage 1; $R = C_1 J_1/C_2 J_2$; $\gamma = T_{10}/T_{20}$; $\gamma_0 = T_0/T_{20}$; $\lambda = \tau_2/\tau_1$; $\xi = z/L$; $\tau = t/t_\alpha$; $t_\alpha = C_s\rho_s/\alpha$. Subscripts: 1, stage of solid-phase heating; 2, stage of gas-phase heating; s, solid phase.

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